Re-exam Waves and Optics – 3 March 2015 – 18:30-21:30

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Questions and answers

A few **preliminary remarks**:
- Please answer questions 3, 4 & 5 on another (double) sheet of paper than questions 1, 2 & 3.
- Put your name and student number at the top of all sheets.
- Put your student card at the edge of the desk for checking by the assistants and show it when handing in your exam.

Question 1 (6 points): states of polarization

Consider the following waves propagating in the z-direction:

1. $\vec{\mathbf{E}} = \hat{\mathbf{i}}\, E_0\, cos(\, kz - \omega t\,) + \hat{\mathbf{j}}\, E_0\, cos(\, kz - \omega t + \pi\,)$

2. $\vec{\mathbf{E}} = \hat{\mathbf{i}}\, E_0\, cos(\, \omega t - kz\,) + \hat{\mathbf{j}}\, E_0\, cos(\, \omega t - kz + \pi/2\,)$

$\hat{\mathbf{i}}, \hat{\mathbf{j}}$ are the unit vectors in the *x*- and *y*-directions.

Question:

Describe the state of polarization of these 2 waves (each separately). More specifically: linear, elliptical or circular and, if relevant, direction of polarization, left or right circularly. Draw a figure to illustrate things (and indicate clearly what the figure is showing !).
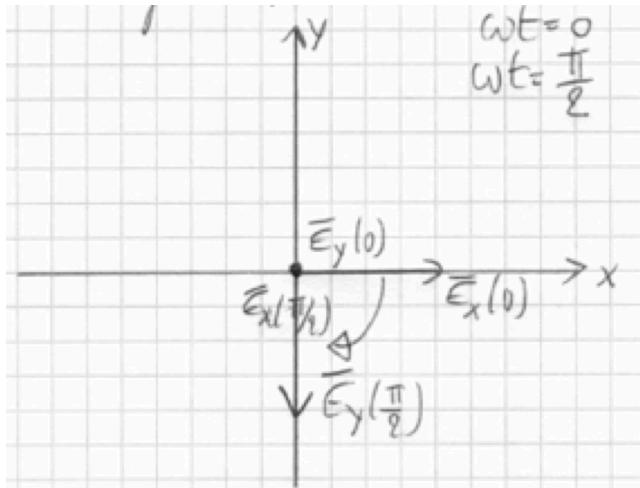
Answer

1.
same amplitude, the components along the *x*- and *y*-directions are π radians out of phase and
$\vec{\mathbf{E}} = \vec{\mathbf{E}}_x + \vec{\mathbf{E}}_y$, so we have a *P*-state (linearly polarised), with the polarisation direction at -45/135 degrees with respect to the *x*-axis

figure for $z = 0$

$\omega t = 0$
$\omega t = \pi$

2.
figure for $z = 0$

$\omega t = 0$
$\omega t = \frac{\pi}{2}$
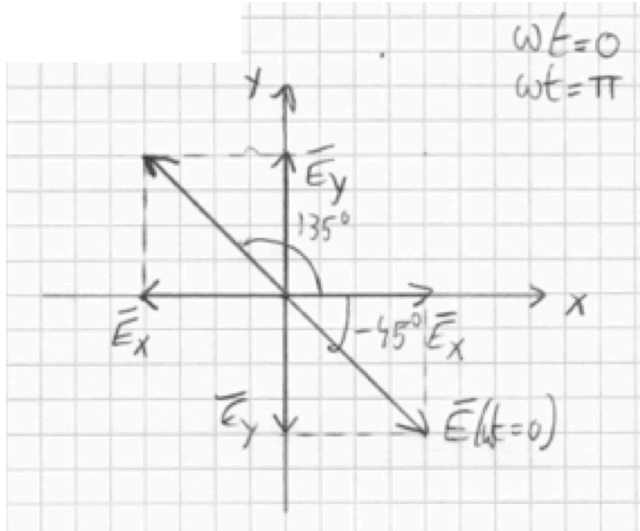
same amplitude, the components along the x- and y-directions are $\pi/2$ radians out of phase and $\vec{\mathbf{E}} = \vec{\mathbf{E}}_x + \vec{\mathbf{E}}_y$, so circularly polarised light

looking against the direction of propagation, $\vec{\mathbf{E}}$ rotates clockwise, so right-circularly polarised light (R)

Question 2 (7 points): Properties of electromagnetic waves

Below Maxwell's equations in vacuum, both in vector form as well as in cartesian coordinates (for (1) and (2) split into x,- y, and z-coordinate).

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1) \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (2)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (1.1) \qquad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \quad (2.1)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad (1.2) \qquad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \quad (2.2)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (1.3) \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t} \quad (2.3)$$

$$\nabla \bullet \vec{E} = 0 \quad (3) \qquad \nabla \bullet \vec{B} = 0 \quad (4)$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad (3.1) \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (4.1)$$

Questions:

Use the example of a plane wave propagating in the positive x-direction (see sketch below) to deduce the following properties of electromagnetic waves:
   a) the electric field ($\vec{E}$) is transversal (perpendicular to (*NL: loodrecht op*) the direction of propagation)
   b) the magnetic field ($\vec{B}$) is transversal (perpendicular to (*NL: loodrecht op*) the direction of propagation)
   c) the electric and magnetic field are perpendicular to each other

 (Use the numbering of the equations above to refer to the equations used.)
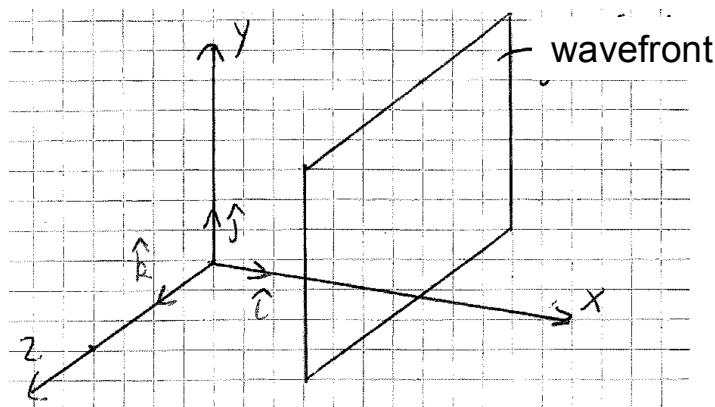
Answer

a)

- we consider a plane wave propagating in the *x*-direction, the wave front is thus $\perp$ *x*-axis
- in all points in the wavefront, the plane wave has the same phase, and thus the same wavefunction $\vec{E}$, so $\dfrac{\partial \vec{E}}{\partial y} = 0$, $\dfrac{\partial \vec{E}}{\partial z} = 0$ and $\vec{E}(x,t)$ ($\vec{E}$ does not depend on *y* and *z*)
- equation (3.1) then becomes $\dfrac{\partial E_x}{\partial x} = 0$ and thus E_x is constant
- a constant E_x does not participate in the wave, so one can put $E_x = 0$ as we are only investigating the wave character of the electric field $\vec{E}$
- if the *x*-component does not participate in the wave, $\vec{E} \perp$ *x*-axis (the direction of propagation by choice) and the electric field is transversal

b)

- from the above follows that $\vec{E}$ // (*y,z*) plane
- without loss of generality, we can choose the *y*-axis direction such that $\vec{E}$ // *y*-axis: $\vec{E} = E_y(x,t)\hat{j}$ ($\hat{j}$ is the unit vector along the *y*-axis)
- equations (1.1), (1.2) and (1.3) then become:

$$0 = -\frac{\partial B_x}{\partial t}$$

$$0 = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

from which we conclude that $\vec{B}$ has a wave character only in the *z*-direction (the *x*- and *y*-components are constant in time and do thus not participate in the wave character)

- thus $\vec{B} \perp$ to the *x*-axis (the direction of propagation by choice) and the magnetic field is transversal

c)
from a) and b) follows that if $\vec{E}$ // *y*-axis, then $\vec{B}$ // *z*-axis, so $\vec{E} \perp \vec{B}$

Question 3 (5 points): Relation Fraunhofer diffraction & Fourier transform

In case of Fraunhofer diffraction at an aperture (see figure below) with constant source strength per unit area (ε_A), the amplitude of the electric field in the point P is given by:

$$E(Y,Z) = \frac{\varepsilon_A \, e^{i(\omega t - kR)}}{R} \iint\limits_{\text{opening}} e^{ik(Yy + Zz)/R} \; dy \; dz$$

The meaning of the coordinates y,z and Y,Z is clear from the figure. R is the distance from the point P to the origin of the coordinate system.

In a more general case where the source strength per unit area is not constant, $\varepsilon_A(y,z)$, the aperture function $A(y,z)$ is defined.
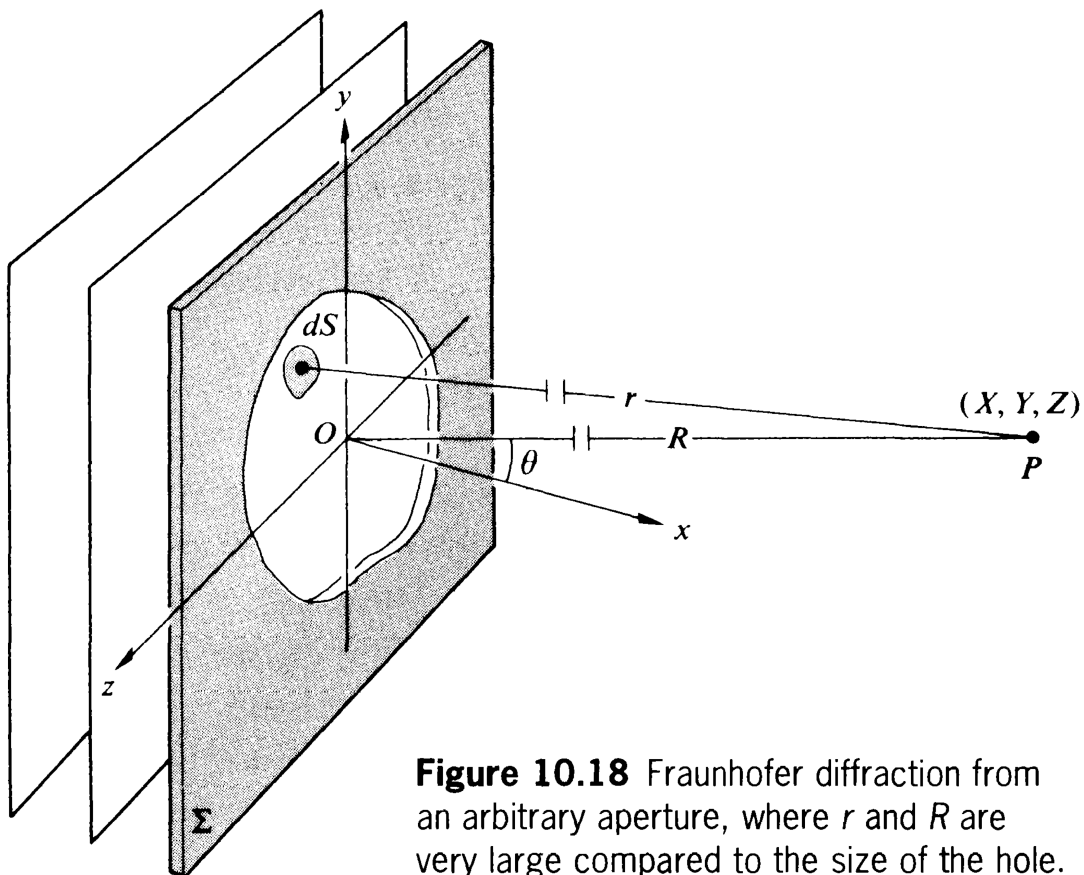

Figure 10.18 Fraunhofer diffraction from an arbitrary aperture, where r and R are very large compared to the size of the hole.

Questions:

1. Give the mathematical expression for the definition of the aperture function.

2. Show the relationship between the Fraunhofer diffraction pattern and the 2-dimensional Fourier transform.

The 2-dimensional Fourier transform of a function dependent on 2 variabeles is given by:

$$\mathscr{F}(f(y,z)) \equiv F(k_y, k_z) = \iint\limits_{-\infty}^{+\infty} f(y,z) \, e^{i(k_y y + k_z z)} dy \; dz$$

Answer:

for ε_A not constant:

$$E(Y,Z) = \iint\limits_{\text{opening}} \frac{\varepsilon_A(y,z)\,e^{i(\omega t - kR)}}{R}\,e^{ik(Yy+Zz)/R}\;dy\;dz$$

1. the aperture function is defined as: $A(y,z) \equiv \dfrac{\varepsilon_A(y,z)\,e^{i(\omega t - kR)}}{R}$

2.

the relationship with the 2D Fourier transform is established by defining the spatial frequencies as follows:

$$k_y \equiv \frac{kY}{R},\; k_z \equiv \frac{kZ}{R}$$

E can then be written as function of k_y, k_z :

$$E(k_y,k_z) = \iint\limits_{-\infty}^{+\infty} A(y,z)\,e^{i(k_y y + k_z z)}dy\;dz \qquad (1)$$

note that the integration limits can be $\pm\infty$ as $A(y,z)$ is zero outside op the aperture

Comparing (1) with the definition of the 2D Fourier transform shows that the distribution of the electric field of a Fraunhofer diffraction pattern is equal to the Fourier transform of the aperture function.

Question 4 (6 points): Refraction of light

A very narrow beam of white light is incident at an angle of 60 degrees on a sheet of glass in air. The index of refraction of the glass for red light is 1.505 and for violet light it is 1.545. The distance between the red and violet light when exiting the sheet of glass is 1 mm.

Question:

What is the thickness of the sheet of glass ? Provide a sketch of the situation to support your answer.

Answer
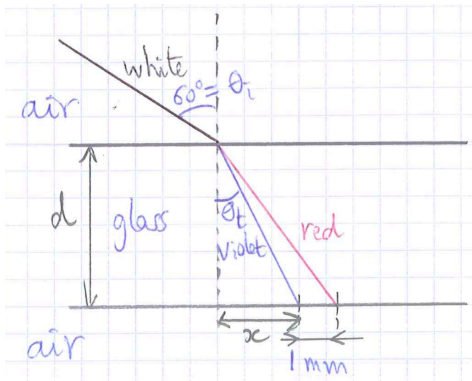
NOTE: one can also consider that the perpendicular distance between the outgoing red and violet beams is 1 mm; this interpretation has been considered to be correct as well

from geometry: $x = d \ tan\theta_t$ (1)

law of refraction (Snell's law): $n_i \ sin\theta_i = n_t \ sin\theta_t$ or $\theta_t = sin^{-1}\left(\dfrac{n_i}{n_t} sin\theta_i\right)$ (2)

combining (1) and (2) gives: $x = d \ tan\left[sin^{-1}\left(\dfrac{n_i}{n_t} sin\theta_i\right)\right]$

the distance between the red and violet light when exiting the sheet of glass is thus

$$x_r - x_v = d \left\{tan\left[sin^{-1}\left(\frac{n_i}{n_{t,r}} sin\theta_i\right)\right] - tan\left[sin^{-1}\left(\frac{n_i}{n_{t,v}} sin\theta_i\right)\right]\right\}$$

filling out $n_i = 1$, $n_{t,r} = 1.505$, $n_{t,v} = 1.545$, $\theta_i = 60°$, $x_r - x_v = 1$ mm and solving for d gives:

$$1 = d \left\{tan\left[sin^{-1}\left(\frac{1}{1.505} sin60\right)\right] - tan\left[sin^{-1}\left(\frac{1}{1.545} sin60\right)\right]\right\} =$$

$$1 = d \left\{tan\left[sin^{-1}(0.5754)\right] - tan\left[sin^{-1}(0.5605)\right]\right\} =$$

$$1 = d \left\{tan[35.13] - tan[34.09]\right\} =$$

$$1 = d \left\{0.7036 - 0.6768\right\} = 0.0268$$

$$d = \frac{1}{0.0268} = 37.3 \text{ mm}$$

Question 5 (6 points): Interference from thin films

White light impinges perpendicularly on a thin oil film (index of refraction 1.30) on a glass substrate (index of refraction 1.50). When looking at the reflected light as a function of wavelength, a minimum irradiance is observed at a wavelength of both 550 and 750 nm.

Questions:

1. Which are the smallest orders of interference minima that are involved ?
2. How thick is the oil film ?

Answer

1. We need to consider the superposition of the light reflected from the front and back of the film. Both reflections are of the same type (external reflection in this case), so there is no relative phase difference due to reflection.

The irradiance of the reflected light will be minimal if the two reflected waves interfere destructively, i.e. when they have an optical path length difference of $\left(m+\frac{1}{2}\right)\lambda$, $m = 0, \pm 1, \pm 2, ...$

The optical path length difference between the light reflected from the front and back of the film is $2nd$ with d the thickness and n the index of refraction of the film. So the condition

$2nd = \left(m+\frac{1}{2}\right)\lambda$ or $d = \left(m+\frac{1}{2}\right)\dfrac{\lambda}{2n}$ must be fulfilled for λ_1=550 and λ_2=750 nm:

$\left(m+\frac{1}{2}\right)\dfrac{\lambda_1}{2n} = \left(m+\frac{1}{2}\right)\dfrac{\lambda_2}{2n}$.

Solution method a: from the above, filling out the numbers, follows: $m_1 = 1.3636m_2 + 0.1818$. The smallest integers fulfilling this equation are: $m_1 = 7$, $m_2 = 5$ So the film thickness is:

$d = \dfrac{7.5 \times 550}{2 \times 1.30} = 1587$ nm or $d = \dfrac{5.5 \times 750}{2 \times 1.30} = 1587$ nm

Solution method b: calculate d for both wavelengths and different orders:

order	thickness oil film	
	550 nm	750 nm
4	952	1298
5	1164	**1587**
6	1375	1875
7	**1587**	2163
8	1798	2452

The same thickness, 1587 nm, follows for $m_1 = 7$, $m_2 = 5$.

Question 6 (5 points): Fraunhofer diffraction of a single slit

The Fraunhofer diffraction pattern of a single slit with width b is given by

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \text{, with } \beta = \frac{k\,b}{2}\,\sin\theta$$

θ is the angle with respect to the center of the slit and k is the propagation number of the light used.

Question

When using light with a wavelength of 450 nm, what is the slit width that will produce a central maximum of the diffraction pattern that has an angular breadth (*NL "breedte"*) of 30 degrees ?
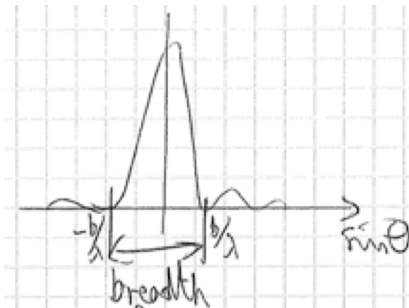
Answer

The breath of the central maximum of the diffraction pattern is determined by the first minima left and right of the central maximum. $I(\theta)$ is mimimal (and equal to 0) for $\sin\beta = 0$, except when $\beta = 0$ $\left(\frac{\sin 0}{0} = 1\right)$, so for $\beta = \pm m\pi$, $m \ge 1$ (m integer). The first minima occur for $m = \pm 1$, $\beta = \pm\pi$.

Using $k = \frac{2\pi}{\lambda}$ this becomes $\pm\pi = \frac{\pi b}{\lambda}\sin\theta_1$ (θ_1 is the angle at which the first minima occur) or

$\pm\sin\theta_1 = \frac{\lambda}{b}$. The situation is sketched below:

For an angular breadth of 30 degrees, the half breadth is 15 degrees. So $\sin 15 = \frac{450 \text{ nm}}{b}$ or

$b = \frac{450 \text{ nm}}{\sin 15} = 1739 \text{ nm} = 1.74 \ \mu m$